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Birzeit University-Deptartment of Mathematics
Maec330-Microeconomic Theory

Midterm Exam
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Question 1 Find the extreme values of the function $f(x, y, z) = x + y + z$ subject to $x^2 + y^2 = 1$ and $x - z = 1$.

$f(x,y,z) = x + y + z$ *equality constraint.

$h_1(x) = x^2 + y^2 = 1$

$h_2(x) = x - z = 1$.

$L(x, y, \lambda_1, \lambda_1) = x + y + z - \lambda_1(x^2 + y^2 - 1) - \lambda_2(x - z - 1)$

① $\frac{dL}{dx} = 1 - 2\lambda_1 x - \lambda_2 = 0$

② $\frac{dL}{dy} = 1 - 2\lambda_1 y = 0$

③ $\frac{dL}{d\lambda_1} = x^2 + y^2 = 1$

④ $\frac{dL}{d\lambda_2} = x - z = 1$

⑤ $\frac{dL}{dz} = 1 + \lambda_2 = 0$.

$\boxed{\lambda_2 = -1}$

④: $z = x - 1$ ... *

③: $y = \sqrt{1 - x^2}$

in ②: $1 - 2\lambda_1 y = 0$

$1 = 2\lambda_1 y$

$1 = 2\lambda_1(\sqrt{1 - x^2})$

$\lambda_1 = \frac{1}{2\sqrt{1-x^2}}$

In ①:

$1 = \cancel{2} \cdot \frac{1}{\cancel{2}\sqrt{1-x^2}} \cdot x + (-1)$

$2\sqrt{1 - x^2} = 1$

$1 - x^2 = \left(\frac{1}{2}\right)^2$

$\pm 1 - x^2 = \frac{1}{4} - 1$

$-(-x^2 = \frac{1}{4} - 1)$

$x^2 = \frac{4}{4} - \frac{1}{4} \Rightarrow x^2 = \frac{3}{4}$

$x^2 = \frac{3}{4} \Rightarrow x = \pm\frac{\sqrt{3}}{2}$ $\lambda_2 = -$

when $x = \pm\frac{\sqrt{3}}{2} \Rightarrow \boxed{x^2}$

$x^2 + y^2 = 1 \Rightarrow \frac{3}{4} + y^2 = 1$

$\frac{4}{4} - \frac{3}{4} = y^2 \Rightarrow y^2 = \frac{1}{4} \Rightarrow \boxed{y = \pm\frac{1}{2}}$

$\lambda_2 = -1$, $x = \pm\frac{\sqrt{3}}{2}$, $y = \pm\frac{1}{2}$

When $y = \frac{1}{2}$.

$1 - 2\lambda_1 y = 0$

$1 - \cancel{2}\lambda_1 \frac{1}{\cancel{2}} = 0$

$\boxed{1 = \lambda_1}$

When $y = -\frac{1}{2}$.

$1 - \cancel{2}\lambda_1 \cdot \frac{-1}{\cancel{2}} = 0$

$\boxed{-1 = \lambda_1}$

When $x = \frac{\sqrt{3}}{2}$

$x - z = 1$

$\frac{\sqrt{3}}{2} - z = 1$

$z = \frac{\sqrt{3}}{2} - \frac{2}{2} = \boxed{\frac{\sqrt{3}-2}{2} = z}$

When $x = -\frac{\sqrt{3}}{2}$

$-\frac{\sqrt{3}}{2} - z = 1$

$\frac{2}{2} - \frac{\sqrt{3}}{2} = z$

$\boxed{z = \frac{2-\sqrt{3}}{2}}$

*Points:

$(x, y, z, \lambda_1, \lambda_2)$

$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}-2}{2}, 1, -1\right)$

$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \frac{2-\sqrt{3}}{2}, -1, -1\right)$

$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}, \frac{\sqrt{3}-2}{2}, -1, -1\right)$

$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{2-\sqrt{3}}{2}, 1, -1\right)$.

$f(x, y, z) = x + y + z$.

$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}-2}{2}\right)$

$f(x, y, z) = \frac{\sqrt{3}}{2} + \frac{1}{2} + \frac{\sqrt{3}-2}{2}$

$= \frac{\sqrt{3} + 1 + \sqrt{3} - 2}{2}$

$= \frac{2\sqrt{3} - 1}{2} = 1.23$

$f\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}, \frac{\sqrt{3}-2}{2}\right)$

$= \frac{\sqrt{3}}{2} - \frac{1}{2} + \frac{\sqrt{3}-2}{2} = \frac{\sqrt{3} - 1 + \sqrt{3} - 2}{2}$

$= \frac{2\sqrt{3} - 3}{2} = 0.23$

$f\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 2 - \frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2} - \frac{1}{2} + \frac{2-\sqrt{3}}{2}$

$= \frac{-\sqrt{3} - 1 + 2 - \sqrt{3}}{2} = \frac{-2\sqrt{3} + 1}{2} = \frac{1 - 2\sqrt{3}}{2} =$

$f\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{2-\sqrt{3}}{2}\right) = \frac{-\sqrt{3} + 1 + 2 - \sqrt{3}}{2}$

$= \frac{-2\sqrt{3} + 3}{2} = \frac{3}{2} - \sqrt{3} = -0.23$

extreme value:-

1- Max at $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}-2}{2}\right)$

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Question 2 Find the maximum of the function $f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$ subject to the constraint $\frac{1}{2}x^2 + y^2 \leq 1$.

$$f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$$
$$h(x) = \frac{1}{2}x^2 + y^2 \leq 1$$

$$L(x) = \frac{1}{2}x^2 + \frac{1}{2}y^2 - \lambda\left(\frac{1}{2}x^2 + y^2 - 1\right)$$

① $\frac{dL}{dx} = x - \lambda x = 0$
$x(1-\lambda) = 0$

② $\frac{dL}{dy} = y - 2\lambda y = 0$

③ $\lambda \geq 0$

④ $\lambda\left(\frac{1}{2}x^2 + y^2 - 1\right) = 0$

⑤ $\frac{1}{2}x^2 + y^2 - 1 \leq 1$.

$x(1-\lambda) = 0$

when $x = 0$

$\lambda\left(\frac{1}{2}(0) + y^2 - 1\right) = 0$

$\lambda y^2 = 1$

$\lambda = \frac{1}{y^2}$

$y - 2 \cdot \frac{1}{y^2} y = 0$

$y - \frac{2}{y} = 0$

$y = \frac{2}{y} \Rightarrow y^2 = 2$

$y = \pm\sqrt{2}$

$\sqrt{2}$: $\lambda = \frac{1}{2}$

$= -\sqrt{2}$: $\lambda = \frac{1}{2}$

in Points: $(0, \sqrt{2}, \frac{1}{2})$
$(0, -\sqrt{2}, \frac{1}{2})$

$f(0, \pm 1)$

when $1 - \lambda = 0$

$\boxed{\lambda = 1}$

~~$y - 2(1)y = 0$~~

~~$y - 2y = 0$~~

$y - 2(1)y = 0$

$y - 2y = 0$

$-y = 0$

$\boxed{y = 0}$

$1\left(\frac{1}{2}x^2 + y^2 - 1\right) = 0$

~~$\frac{1}{2}x^2 + y^2 - 1 = 0$~~

~~$\frac{1}{2}x^2 + y^2 = 1$~~

~~$x^2 + 2y^2 = 2$~~

~~$x = \pm\sqrt{2 - 2y^2}$~~

~~$x = \sqrt{2 - 2y^2}$~~

$1\left(\frac{1}{2}x^2 + 0 - 1\right) = 0$

$\frac{1}{2}x^2 - 1 = 0$

$\frac{1}{2}x^2 = 1$

$x^2 = 2$

$x = \pm\sqrt{2}$.

$(\sqrt{2}, 0, 1)$
$(-\sqrt{2}, 0, 1)$.

when $\lambda = 0$
$x = 0$, $y = 0$
$(0, 0, 0)$

$f(0, \sqrt{2}) = 1$

$f(0, -\sqrt{2}) = 1$

$f(\sqrt{2}, 0) = 1$

$f(-\sqrt{2}, 0) = 1$

$f(0, 0) = 0$

$\therefore$ the max point is $(0, \pm\sqrt{2})$
& $(\pm\sqrt{2}, 0)$

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Question 3 Find the minimum of the function $f(x,y) = x^2 + y^2 - x - y$ subject to $x^2 + y^2 \le a$, $x \ge 0, y \ge 0$, where a is a positive real number.

$f(x,y) = x^2 + y^2 - x - y$.

$h_1(x) \Rightarrow x^2 + y^2 \le a$.

$h_2(x) \Rightarrow -x \le 0$

$h_3(x) \Rightarrow -y \le 0$

$-x^2 - y^2 \ge -a$

$L(x, y, \lambda_1, \dots \lambda_n) = x^2 + y^2 - x - y - \lambda_1 (x^2 + y^2 - a) + \lambda_2 x + \lambda_3 y$.

① $\frac{\partial L}{\partial x} = 2x - 1 - 2\lambda_1 x + \lambda_2 = 0$

② $\frac{\partial L}{\partial y} = 2y - 1 - 2\lambda_1 y + \lambda_3 = 0$

③ $\lambda_1 (x^2 + y^2 - a) = 0$

④ $\lambda_2 x = 0$

⑤ $\lambda_3 y = 0$

⑥ $\lambda_2 \ge 0$, ⑧ $\lambda_1 \ge 0$, ⑧ $\lambda_3 \ge 0$

⑦ $x \ge 0$, $y \ge 0$

~~2x - 1 - 2~~

$\lambda_1 (x^2 + y^2 - a) = 0$

$\lambda_1 = 0 \quad \wedge \quad x^2 + y^2 = a$.

$\lambda_2 = 0$

$\lambda_3 = 0$

$2x - 1 - 0 + 0 = 0$

$2x = 1$

$\boxed{x = \frac{1}{2}}$

$2y - 1 - 0 + 0 = 0$

$\boxed{y = \frac{1}{2}}$

$(x, y, \lambda_1, \lambda_2, \lambda_3)$

$P\left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$.

$x = 0$

$y = 0$

$\lambda_2 = 1 \rightarrow 0 = 0 - 1 - 0 +$

$\lambda_1 = 0$

$\lambda_3 = 1$

$(0, 0, 0, 1, 1)$.

$f(0, 0) = 0$

$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} - \frac{1}{2} - \frac{1}{2}$

$= \frac{2}{4} - \frac{2}{2} = \frac{1}{2} - 1$

$= -\frac{1}{2}$.

$\therefore$ min Point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

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Question 4 Suppose that the point $(x, y, \mu) = (1, 4, 4)$ is a critical point of the Lagrangean function $L(x, y, \mu) = 2x + y - \mu(\sqrt{x} + \sqrt{y} - 1)$.

(a) Use the second order conditions to show that the given point is a minimizer of $f(x, y) = 2x + y$ subject to $\sqrt{x} + \sqrt{y} = 1$.

$L = 2x + y - \lambda(\sqrt{x} + \sqrt{y} - 1)$

$h(x) = \sqrt{x} + \sqrt{y} = 1$

$f(x,y) = 2x + y$

$\frac{df}{dx} = 2$, $\frac{df}{dy} = 1$, $\frac{dh}{dx} = \frac{1}{2\sqrt{x}}$, $\frac{dh}{dy} = \frac{1}{2\sqrt{y}}$

$$H = \begin{pmatrix} 0 & \frac{dh}{dx} & \frac{dh}{dy} \\ \frac{dh}{dx} & \frac{d^2L}{dx^2} & \frac{d^2L}{dxdy} \\ \frac{dh}{dy} & \frac{d^2L}{dxdy} & \frac{d^2L}{dy^2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2\sqrt{x}} & \frac{1}{2\sqrt{y}} \\ \frac{1}{2\sqrt{x}} & \frac{-\lambda}{\sqrt{x^3}} & \\ \frac{1}{2\sqrt{y}} & & \frac{-\lambda}{\sqrt{y^3}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -4 & 0 \\ \frac{1}{4} & 0 & -2 \end{pmatrix}$$

$\det H < 0 \Rightarrow$

$\det H = +\frac{1}{2}(-1) + \frac{1}{4}(1) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} < 0$

$\therefore \det H < 0$

$\therefore \exists$ min points

(b) Use the envelope theorem to estimate the minimum of f subject to the constraint $0.9\sqrt{x} + \sqrt{y} = 1$.

$L = 2x + y - \lambda(a\sqrt{x} + \sqrt{y} - 1)$

$f^*_{new} = f^*_{old} + \frac{dL}{da} \cdot da$

$\frac{dL}{da} = -\lambda\sqrt{x} \Big|_{\lambda = 4,\ x = 1} = -4\sqrt{1} = -4$

$= 6 + (-4)(-0.1)$

$= 6 + \frac{4}{10}$

$f^*_{new} = \frac{60 + 4}{10} = \frac{64}{10}$

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Question 5 Answer the following:

1. Determine whether the following functions are homogeneous, homothetic or neither.

   (a) $f(x,y) = \sin(y/x) + \cos(x/y)$.

$$f(tx,ty) = \sin\left(\frac{yt}{xt}\right) + \cos\left(\frac{xt}{yt}\right)$$

$$= \sin\left(\frac{y}{x}\right) + \cos\left(\frac{x}{y}\right)$$

$$= t^0 f(x,y)$$

$\therefore$ its homogenous of degree $k=0$.

   (b) $f(x,y) = x^4y^2 + x^2y + 1$. >0

Not Homogenouse. , but Homothetic.

$$\frac{df}{dx} = 4x^3y^2 + 2xy > 0 \Rightarrow \frac{d^2f}{dx^2}$$

$$\frac{df}{dy} = 2yx^4 + x^2 > 0$$

2. Show that if $f(x,y)$ is homogeneous of degree k then

$$x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = k(k-1)f.$$

~~$f(x,y) = f(t^k x, t^k y) = t f(x,y)$~~

~~$\frac{df}{dx} = t^{k-1} f'(x) \Rightarrow$~~

$$f(tx,ty) = t^k f(x,y) \Rightarrow f(t^k x, t^k y)$$

$$\frac{df}{dy} = f(t^k x, t^k y) \Rightarrow \frac{d^2f}{dy^2} f(t^k x, 0)$$

$$\frac{df}{dx} = f(t^k, t^k y) \Rightarrow \frac{d^2f}{dx^2} = f(0, t^k y)$$

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$$\boxed{x_1 \, df x_1 + y \, df_y = kf}$$

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